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Equivalence of utilitarian maximal and weakly maximal programs

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ABSTRACT

For a class of aggregative optimal growth models, which allow for a non-convex and non-differentiable production technology, this paper examines whether the set of utilitarian maximal programs coincides with the set of weakly maximal programs. It identifies a condition, called the Phelps–Koopmans condition, under which the equivalence result holds. An example is provided to demonstrate that the equivalence result is invalid when the Phelps–Koopmans condition does not hold.

1. Introduction

In a recent paper, Basu and Mitra (2007) proposed a new utilitarian criterion[1] for evaluating infinite utility streams. They argue that the axiomatic basis of their utilitarian criterion is more compelling than that of the overtaking or the catching up criteria, used by Ramsey (1928); Atsumi (1965); von Weizšacker (1965); Gale (1967); Brock (1970a,b).[2] However, the utilitarian criterion is a more incomplete quasi-order.[3] To elucidate that the lack of comparability is not a severe handicap in general, Basu and Mitra (2007) show that for the standard neoclassical aggregative growth model, any "utilitarian maximal" program (maximal in the sense of being undominated in terms of the utilitarian quasi-order by any other feasible program from the same initial stock) *overtakes* all other programs starting from the same initial stock. So, in particular, the set of utilitarian maximal programs is identical to the set of "weakly maximal" programs (Brock, 1970a) from any positive initial stock.

We examine whether this equivalence result holds for a larger class of aggregative optimal growth models, which allow for a non-convex and non-differentiable production technology. This is the main objective of the paper.

Our result characterizes those models where the equivalence result holds and where it fails. One would expect that the set of "maximal programs" obtained for a more incomplete quasi-order (utilitarian criterion) to be larger than the set of maximal programs from the relatively more complete quasi-order (overtaking criterion). Actually the sets turn out to be the

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[1] In this paper, we refer to the criterion simply as "utilitarian". A precise definition in our set up is given in Section 2.1.2.

[2] Some aspects of questionable rankings of the overtaking criterion are also discussed in Asheim and Tungodden (2004).

[3] A quasi-order $\succsim$ on a set X is a reflexive ($x \succsim x$ for all $x \in X$) and transitive (for $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$) binary relation.

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same for an interesting subset of the class of aggregative growth models considered in this paper, although the equivalence result does *not* hold for the entire class. Our analysis identifies a condition (which we call the Phelps–Koopmans condition) which separates models for which the equivalence result holds from those for which it fails.

To elaborate, the Phelps–Koopmans condition states that if the stocks along some feasible program converges to a stock above the minimum golden rule stock, then the program is inefficient. This condition serves as the dividing line for models where any utilitarian maximal program is weakly maximal (and conversely), and where this equivalence fails.

Using the equivalence of utilitarian maximal and weakly maximal programs for models satisfying the Phelps–Koopmans condition, we show that if a program is competitive (so there exists price support for intertemporal utility and profit maximization) and efficient (there being no other program from the same initial stock that gives at least as much consumption in all periods and strictly more in some), then it must be weakly maximal. This generalizes the sufficiency part of a famous characterization theorem of weak maximality due to Brock (1971).

2. Preliminaries

2.1. The model

We present an aggregative growth model where the production function is not necessarily concave or smooth[4] and future utilities are not discounted.

2.1.1. Production

The production technology is summarized by a production function, f, mapping $\mathbb{R}_+$ to itself. The following assumptions are maintained on the function f throughout.

(F.1.). $f(0)=0$; f is increasing and continuous for all $x \geq 0$,

(F.2.). There is some $\bar{k}>0$ such that (i) $f(x)>x$ for all $x<\bar{k}$ and (ii) $f(x)<x$ for all $x>\bar{k}$.

Assumptions (F.1) and (F.2) are standard. Note that (F.2) guarantees the existence of a unique *maximum sustainable stock*, $\bar{k}$.

It can be shown that there is some $k \in (0, \bar{k})$ such that:

$$f(k)-k \geq f(x)-x \quad \text{for all } x \geq 0 \tag{1}$$

Observe that k in (1) need not be unique. Any k satisfying (1) is called a *golden rule* stock. The set of all *golden rule* stocks is denoted by $\mathcal{G}$. By (F.2), $\mathcal{G}$ is a subset of $(0, \bar{k})$. Obviously, for any $k, k' \in \mathcal{G}$

$$f(k)-k=f(k')-k' \tag{2}$$

We denote this (common) value in (2) by c^*. A *program* from (the initial stock) $\mathbf{k} \geq 0$ is a sequence $\langle k_t \rangle$ for all $t \geq 0$ satisfying:

$$k_0=\mathbf{k}; \quad 0 \leq k_t \leq f(k_{t-1}) \quad \text{for all } t \geq 1 \tag{3}$$

The *consumption program* $\langle c_t \rangle$ generated by $\langle k_t \rangle$ is given by

$$c_t=f(k_{t-1})-k_t \quad \text{for all } t \geq 1$$

It is easy to show that under the given restrictions on f, for every feasible program $\langle k_t \rangle$ from $\mathbf{k} \geq 0$

$$k_t \leq B(\mathbf{k}) \quad \text{for all } t \geq 0; \qquad c_t \leq B(\mathbf{k}) \quad \text{for all } t \geq 1 \tag{4}$$

where $B(\mathbf{k})=\max\{\bar{k}, \mathbf{k}\}$. The analysis of the paper will be restricted to the interesting case where the initial stock $\mathbf{k} \in [0, \bar{k}]$. In this case $k_t \leq \bar{k}$ for all $t \geq 0$; $c_t \leq \bar{k}$ for all $t \geq 1$.

A program $\langle k'_t \rangle$ from $\mathbf{k} \geq 0$ *dominates* a program $\langle k_t \rangle$ from $\mathbf{k}$, if $c'_t \geq c_t$ for all $t \geq 1$ and $c'_t > c_t$ for some t. A program $\langle k_t \rangle$ from $\mathbf{k}$ is said to be *inefficient* if some program from $\mathbf{k}$ dominates it. It is said to be *efficient* if it is not inefficient.

2.1.2. Preferences

We let u, a function from $\mathbb{R}_+$ to $\mathbb{R}$, denote the preferences of the social planner. The following assumption on u is maintained throughout.

(U.1.). $u(c)$ is strictly increasing, continuous and strictly concave for $c \geq 0$.

A program $\langle k'_t \rangle$ from $\mathbf{k} \geq 0$ *utilitarian dominates* a program $\langle k_t \rangle$ from $\mathbf{k}$, if there is some $T \in \mathbb{N}$ such that $\sum_{t=1}^{T}(u(c'_t)-u(c_t))>0$ and $u(c'_t) \geq u(c_t)$ for all $t \geq T+1$. A program $\langle k'_t \rangle$ from $\mathbf{k} \geq 0$ is called *utilitarian maximal* if there is no program from $\mathbf{k}$ that utilitarian dominates it.

[4] This class of growth models was studied by Mitra and Ray (1984), but with a discounted utilitarian criterion.

A program $\langle k'_t \rangle$ from $\mathbf{k} \geq 0$ *strongly overtakes* a program $\langle k_t \rangle$ from $\mathbf{k}$ if there exists $\alpha > 0$ and N_0 such that for all $T \geq N_0$,

$$\sum_{t=1}^{T} (u(c'_t) - u(c_t)) \geq \alpha$$

A program $\langle k'_t \rangle$ from $\mathbf{k} \geq 0$ is *weakly maximal* if there is no program from $\mathbf{k}$ that strongly overtakes it. This definition of weak maximality is due to Brock (1970a).

A program $\langle k_t \rangle$ from $\mathbf{k} \geq 0$ is *good* if there exists some $G \in \mathbb{R}$ such that,

$$\sum_{t=1}^{N} (u(c_t) - u(c^*)) \geq G \quad \text{for all} \ \ N \geq 1 \tag{5}$$

A program is called *bad* if

$$\sum_{t=1}^{N} (u(c_t) - u(c^*)) \to -\infty \quad \text{as} \ \ N \to \infty \tag{6}$$

2.2. Competitive programs

In our model, since the production function is not necessarily concave, there might not be dual variables ("competitive" or "shadow" prices) supporting a weakly maximal or utilitarian maximal program. Nevertheless, the notion of programs supported by such dual variables plays an important role in our analysis. In view of this, we introduce here the concept of a competitive program.

A program $\langle k_t \rangle$ from $\mathbf{k}$ is a *competitive program* from $\mathbf{k}$, if there is a sequence $\langle p_t \rangle$ of non-negative numbers, such that for all $t \geq 1$, the following two properties hold:

$$\begin{aligned} &(a) \quad u(c_t) - p_t c_t \geq u(c) - p_t c \quad \text{for all} \ \ c \geq 0 \\ &(b) \quad p_t f(k_t) - p_{t-1} k_{t-1} \geq p_t f(z) - p_{t-1} z \quad \text{for all} \ \ z \geq 0 \end{aligned} \tag{CE}$$

In this case, we refer to the sequence $\langle p_t \rangle$ as *competitive prices* associated with the program $\langle k_t \rangle$.

If $\langle k_t \rangle$ is a *competitive program* from $\mathbf{k}$, with associated competitive prices $\langle p_t \rangle$, and $(k_t, c_t) \gg 0$ for all $t \geq 1$, and f and u are differentiable on $\mathbb{R}_{++}$, then it is also a *Ramsey-Euler program*; that is, it satisfies:

$$u'(c_t) = f'(k_t) u'(c_{t+1}) \quad \text{for all} \ \ t \geq 1 \tag{RE}$$

If $\langle k_t \rangle$ is a program from $\mathbf{k}$, which satisfies $(k_t, c_t) \gg 0$ and (RE) for all $t \geq 1$, and f and u are differentiable on $\mathbb{R}_{++}$, and f and u are concave on $\mathbb{R}_+$, then $\langle k_t \rangle$ is also a *competitive program* from $\mathbf{k}$, with associated competitive prices $\langle p_t \rangle$ given by $p_t = u'(c_t)$ for $t \geq 1$, and $p_0 = p_1 f'(k_0)$.

2.3. Price supported golden rule

In this section, we note the existence of a *stationary* price support of the minimum golden rule capital stock (Proposition 1). The importance of a price supported golden-rule for the theory of optimal growth was recognized by Gale (1967); McKenzie (1968) and Brock (1970a) in models where the technology set is a convex set. It turns out that the concept continues to play a significant role in the theory when the technology set is not a convex set, as demonstrated by Majumdar and Mitra (1982) in the context of an aggregative framework with an S-shaped production function, and by Mitra (1992) in the context of a multisectoral model where the technology set is star-shaped with respect to its golden-rule point.

Let us denote by k^* the smallest golden rule capital stock; that is, $k^* = \min\{s : s \in \mathcal{G}\}$.[5] Positivity of k^* follows from (F.1) and (F.2). Recall that $c^* = f(k^*) - k^*$. Positivity of c^* follows from (F.2).

Proposition 1. *Assume (U.1), (F.1) and (F.2). There is* $p^* > 0$ *such that*

$$u(c^*) - p^* c^* \geq u(c) - p^* c \quad \text{for} \ \ c \geq 0 \tag{UP}$$

$$p^* f(k) - p^* k \geq p^* f(x) - p^* x \quad \text{for} \ \ x \geq 0 \quad \text{and any} \ \ k \in \mathcal{G} \tag{FP}$$

A consequence of Proposition 1 is that if a program $\langle k_t \rangle$ from $\mathbf{k} \in [0, \bar{k}]$ is not good, then it is bad.[6]

[5] The continuity of f guarantees that $\mathcal{G}$ is a closed subset of the compact set $[0, \bar{k}]$. Since k^* is the minimum value attained on a compact set, it is well-defined.

[6] In Corollary 1, (i) can be inferred by using the method used in Majumdar and Mitra (1982, p.116), and (ii) can then be obtained directly from Gale (1967). The proof of Corollary 1 is therefore omitted.

Corollary 1. *Assume (U.1), (F.1) and (F.2). If $\langle k_t \rangle$ is a program from $\mathbf{k} \in [0, \bar{k}]$, then*

(i) $\sum_{t=1}^{N}(u(c_t) - u(c^*)) \le p^*\bar{k}$ *for all* $N \ge 1$; *and*
(ii) *if* $\langle k_t \rangle$ *is not good, then it is bad.*

3. The equivalence result

In this section, we present the principal equivalence result of the paper, which identifies a class of aggregative growth models for which the set of utilitarian maximal programs coincides with the set of weakly maximal programs.

3.1. Preliminary discussion

It follows from the definitions of Section 2.1.2 that weakly maximal programs are necessarily utilitarian maximal. Thus, in establishing our equivalence result, we focus on the conditions under which every utilitarian maximal program is weakly maximal.

Basu and Mitra (2007) showed that in the standard aggregative neoclassical model, with strictly concave and smooth production and utility functions, every utilitarian maximal program is necessarily weakly maximal. The demonstration of this result rests on Brock's (1971) characterization of weakly maximal programs as the class of Ramsey–Euler programs which are efficient. Since one can provide a more acceptable axiomatic basis for the utilitarian quasi-ordering than the overtaking quasi-ordering, this means that at least for a class of important growth models, the use of the more restrictive overtaking quasi-ordering is superfluous.

In seeking to extend the Basu–Mitra observation to aggregative models with non-concavities in the production function, one runs into the difficulty that Brock's characterization result is no longer valid. In terms of his demonstration of the characterization result, the failure occurs at two levels. One arises from the well-known fact that (even with smooth u and f), a Ramsey–Euler program is not necessarily competitive, when f exhibits non-concavities. The other and more subtle failure arises from the observation that (when f exhibits non-concavities) a competitive program, which is efficient, need not be weakly-maximal. [We will return to this last observation in Sections 4 and 5.]

It is clear, then, that we need a new approach. This approach rests on two observations regarding the properties of utilitarian maximal programs (starting from positive initial stocks). Utilitarian maximal programs are efficient and they are good. The first property follows trivially from the definitions (since u is increasing). The second property is non-trivial, and we discuss and establish it in the next section.

In order to build on the second property, one would like to show the "turnpike property" that the stock levels along a good program converge to *some* golden-rule stock. However, even though strict concavity of u ensures that consumption levels along a good program converge to the golden-rule consumption, the convergence of stocks to some golden-rule stock does not follow. We establish the convergence of stocks under the condition that the set of golden-rule stocks has finite cardinality (see condition **(G)** in Section 3.3). Under this additional condition, the stock levels along a utilitarian maximal program do converge to some golden-rule stock.

When f is concave, the first property (efficiency of the utilitarian maximal program) would in fact ensure that the stock levels along a utilitarian maximal program converge to the *minimal* golden-rule stock, because of the Phelps–Koopmans theorem.[7] However, this theorem is not valid in general for non-concave f (see Mitra and Ray, 2009). So, we *impose the condition* that all programs converging to golden-rule stocks above the minimal golden-rule stock are inefficient (we call this the Phelps–Koopmans condition). Clearly, under this condition, the stock levels along every utilitarian maximal program must converge to the minimal golden-rule stock. It is then important to know the technological restrictions for non-concave f, which ensure that the Phelps–Koopmans condition holds. These are provided by Mitra and Ray (2009) and are discussed briefly in Section 5.

The results summarized above help us to establish the equivalence result. In putting together these ingredients to arrive at the desired result, the role of the property that the stock levels along a utilitarian maximal program $\langle k_t \rangle$ from $\mathbf{k} \in (0, \bar{k}]$ converge to the *minimal* golden-rule stock, k^*, becomes clear. It allows one to follow any good program $\langle k'_t \rangle$ from $\mathbf{k}$ for a long enough finite time period, and then switch to the program $\langle k_t \rangle$ with as small a loss in utility as one wishes in making the switch. The utilitarian maximal program $\langle k_t \rangle$ can then be shown to be weakly maximal since it must have at least as large a utility sum over the finite time period (including the switch) compared to any such good comparison program $\langle k'_t \rangle$.

[7] This result was conjectured by Phelps (1962) and proved in Phelps (1965), using an idea suggested by Koopmans. It states that if the capital stock accumulated along a program is above and is bounded away from the golden rule capital stock, then such a program must be inefficient.
The validity of the Phelps–Koopmans theorem for concave f does not depend on Condition **(G)**. The careful reader will no doubt observe that if f is concave and **(G)** holds, then there is actually a unique golden-rule stock. Even though this scenario is somewhat restrictive, it still encompasses the class of growth models considered by Basu and Mitra (2007).

3.2. Utilitarian maximal programs are good

If there is a good program from an initial stock, then any weakly maximal program from that stock must be good, in view of Corollary 1. Any utilitarian maximal program from that stock also has this property, but it does not follow as directly and in fact is one of the key steps in establishing the equivalence result.

What does follow quite directly is that there is $k' \in (k^*, \bar{k})$ such that if $\langle k_t \rangle$ is any utilitarian maximal program from $\mathbf{k} \in (0, \bar{k}]$, then there is a subsequence $\{t_s\}$ of time periods for which $k_{t_s} \leq k'$. And this enables one to construct a sequence of programs (indexed by s) from $\mathbf{k}$ such that for all s large, (a) program s coincides with $\langle k_t \rangle$ for all $t > t_s$, and (b) each program stays at the minimum golden-rule stock for all but a fixed finite number of periods. This enables one to infer that $\langle k_t \rangle$ must be good. We state the result here; the proof (which fills in the details in the outline provided above) is presented in Section 6.

Theorem 1. *Assume (U.1), (F.1) and (F.2). If $\left\langle k_t \right\rangle$ is a utilitarian maximal program from some $\mathbf{k} \in (0, \bar{k}]$, then it is good.*

3.3. A turnpike property of good programs

The price-support property of the minimum golden-rule stock, noted in Proposition 1, entails that the "value-loss lemma" of Radner (1961), as modified for Ramsey-optimal growth models by Atsumi (1965); Gale (1967) and McKenzie (1968), remains in full force even though the production set is non-convex. [This was noted, and fully exploited, in Majumdar and Mitra (1982).] A consequence is that any program $\langle k_t \rangle$ suffers "value-losses" (at the supporting price p^*) if $[f(k_t) - k_t]$ is different from $[f(k^*) - k^*] \equiv c^*$, or if c_t is different from the golden-rule consumption c^*, the value losses being uniform when the differences are uniform. For any *good* program $\langle k_t \rangle$, it is straightforward to see that the sum of these value-losses cannot become infinitely large. That is, for any good program $\langle k_t \rangle$, one must have c_t converging to c^* and $[f(k_t) - k_t]$ converging to $[f(k^*) - k^*] \equiv c^*$.

It follows from these observations that if $\langle k_t \rangle$ converges, it must converge to a golden-rule stock. However, it does not follow from these observations that $\langle k_t \rangle$ actually converges. The convergence of $\langle k_t \rangle$ can be ensured under the following condition:

(**G**) The set $\mathcal{G}$ has a finite number of elements.

It is useful to recall at this point that **(G)** clearly holds when there is only one golden-rule stock, as in Majumdar and Mitra (1982), or Basu and Mitra (2007).

Proposition 2. *Assume (U.1), (F.1), (F.2) and (G). If $\left\langle k_t \right\rangle$ is a good program from some $\mathbf{k} \in (0, \bar{k}]$,* then *$k_t \to k$ for some $k \in \mathcal{G}$.*

3.4. Efficiency and the Phelps–Koopmans theorem

Any utilitarian maximal program in our framework is necessarily efficient, since u is increasing. From Theorem 1 and Proposition 2, we also know that it has the property that stocks converge to some golden-rule stock. We want to claim that the stocks must converge to the minimum golden-rule stock.

Golden-rule stocks above the minimum golden-rule stock correspond to *inefficient* stationary programs. However, programs along which the stocks *converge* to such a golden-rule stock need not be inefficient, so that "over-accumulation of capital" need not signal inefficiency; see Mitra and Ray (2009). That is, the well-known Phelps–Koopmans theorem, which is valid for concave production functions, does not extend to the class of models considered here. Thus, we cannot establish our claim by invoking the efficiency property of utilitarian maximal programs.

More can be said. For the class of models considered here (including the restriction **(G)**), it is possible for the stocks along a utilitarian maximal program to converge to a golden-rule stock above the minimum golden-rule stock; for an example, see Section 5.

To establish our claim, we in fact *impose* the condition that all programs converging to stocks above the minimal golden-rule stock are inefficient, and we call this the Phelps–Koopmans condition.[8]

Phelps–Koopmans condition: If $\left\langle k_t \right\rangle$ is a feasible program from some $\boldsymbol{k} \in (0, \bar{k}]$ and satisfies $\lim_{t\to\infty} k_t = \hat{k}$ and $\hat{k} > k^*$, then $\left\langle k_t \right\rangle$ is an inefficient program.

Proposition 3. *Assume (U.1), (F.1), (F.2), (G) and the Phelps–Koopmans condition. If $\left\langle k_t \right\rangle$ is a utilitarian maximal program from $\mathbf{k} \in (0, \bar{k}]$, then $k_t \to k^*$ as $t \to \infty$.*

We now discuss a sufficient condition on f that guarantees the validity of the Phelps–Koopmans condition. When the production function, f, is concave, then the Phelps–Koopmans condition clearly holds, since the Phelps–Koopmans theorem is valid in that framework. It also holds for the S-shaped production function model of fisheries (due to Clark, 1971) studied in detail by Majumdar and Mitra (1982, 1983).

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[8] One might feel that the Phelps–Koopmans condition makes Proposition 3 trivial. It does make its proof trivial, which is therefore omitted. But, identifying this sufficient condition is non-trivial; further, having identified this condition, it is then possible to seek technological conditions under which it is valid.

Mitra and Ray (2009) show that when f is twice continuously differentiable in some open neighborhood of every golden rule with $f''(\hat{k}) < 0$ for all $\hat{k} \in \mathcal{G}$ and:

$$[-f''(k^*)] < [-f''(\hat{k})] \quad \text{for every} \quad \hat{k} \in \mathcal{G} \quad \text{with} \quad \hat{k} > k^* \tag{F}$$

then the Phelps–Koopmans condition also holds.[9] The sufficient condition (F) is useful in the context of a non-concave production function, since it can be checked with local information about such a function at its golden-rule stocks.[10]

Intuitively, condition (F) states that the marginal product of capital is less sensitive to changes in input variations in a neighborhood of the minimum golden rule when compared to input variations in some neighborhood of the other golden rule stocks. If a program converges to a golden rule higher than the minimum golden rule, the local information in (F) allows one to follow an alternative program which gets close to the minimum golden rule and maintain as much consumption as the original program in all periods and strictly more in some, thereby establishing the inefficiency of the original program. The precise details are spelled out in Proposition 2 of Mitra and Ray (2009).

3.5. *Utilitarian maximality and weak maximality*

It can now be established that a utilitarian maximal program $\langle k_t \rangle$ must be weakly maximal. Otherwise, there would be a program $\langle k'_t \rangle$ from the same initial stock which strongly overtakes $\langle k_t \rangle$. Since $\langle k_t \rangle$ is good, this makes $\langle k'_t \rangle$ good as well, so that by the turnpike property for good programs, $\langle k'_t \rangle$ must converge to some golden-rule stock. By Proposition 3, $\langle k_t \rangle$ must converge to the minimum golden rule stock. It is now possible to see that by following the program $\langle k'_t \rangle$ for a long enough time period (to allow both $\langle k_t \rangle$ and $\langle k'_t \rangle$ to get sufficiently close to their respective limits) and then switching to $\langle k_t \rangle$ beyond that would produce a program which utilitarian dominates $\langle k_t \rangle$, contradicting the utilitarian maximality of $\langle k_t \rangle$.

Theorem 2. *Assume (U.1), (F.1), (F.2), (G) and the Phelps–Koopmans condition. Then,* $\left\langle k_t \right\rangle$ *is a utilitarian maximal program from* $\mathbf{k} \in (0, \bar{k}]$ iff $\left\langle k_t \right\rangle$ *is a weakly maximal program from* $\mathbf{k}$.

4. On a characterization of utilitarian maximal programs

For competitive programs, efficiency is equivalent to weak-maximality when the production function, f, is concave. This result of Brock (1971) fails to hold when f is not concave (as will be clear from the example presented in Section 5). Thus, it is of interest to note that even when f is not necessarily concave, efficiency is equivalent to utilitarian maximality for competitive programs.

Of course, when f is not concave then utilitarian maximal or weakly maximal programs need not be competitive. So, it is useful to provide a more basic characterization result of utilitarian maximal programs in terms of *short-run optimality* and efficiency, from which the result stated in the above paragraph follows.

A program $\langle \hat{k}_t \rangle$ from $\mathbf{k} \in (0, \bar{k}]$ is *short-run optimal* if for every $T \in \mathbb{N}$, $(k_0,\ldots,k_T) = (\hat{k}_0, \ldots, \hat{k}_T)$ solves the problem:

$$\left.\begin{array}{lll} \textit{Max} & \displaystyle\sum_{t=0}^{T-1} u(f(k_t) - k_{t+1}) & \\ \textit{subject to} & 0 \le k_{t+1} \le f(k_t) & \text{for } t = 0, \ldots, T-1 \\ \textit{and} & k_0 = \mathbf{k}, k_T \ge \hat{k}_T & \end{array}\right\} (P)$$

That is, a program $\langle \hat{k}_t \rangle$ is short-run optimal if it is finite-horizon optimal (with terminal stock at least as large as that for the program $\langle \hat{k}_t \rangle$ for that horizon) for *every* finite horizon.

We can now state the following characterization of utilitarian maximal programs. The proof, being entirely straightforward, is omitted.

Theorem 3. *Assume (U.1), (F.1) and (F.2). Let* $\langle \hat{k}_t \rangle$ *be a program from* $\mathbf{k} \in (0, \bar{k}]$. *Then* $\langle \hat{k}_t \rangle$ *is utilitarian maximal if and only if (i) it is short-run optimal, and (ii) it is efficient.*

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[9] Our version of the Phelps–Koopmans condition is "Phelps–Koopmans version II" in Mitra and Ray (2009). It should be noted that the class of production functions satisfying (s1) cannot be concave, when there are multiple golden rule stocks.

[10] Mitra and Ray (2009) obtain an "almost" complete characterization of the Phelps–Koopmans condition in aggregative models where the production function is not necessarily concave and has a finite number of golden rule stocks. In addition to the result stated in the text, they show that when $[-f''(k^*)] > [-f''(\hat{k})]$ for some $\hat{k} \in \mathcal{G}$ with $\hat{k} > k^*$, then the Phelps–Koopmans condition *fails*.

If $\langle \hat{k}_t \rangle$ is a competitive program from $\mathbf{k} \in (0, \bar{k}]$, with associated prices $\langle \hat{p}_t \rangle$, then, for every $T \in \mathbb{N}$, and $(k_0, \ldots, k_T)$ satisfying the constraints of problem (P), denoting $[f(k_t) - k_{t+1}]$ by c_{t+1} for $t = 0, \ldots, T-1$, we have:

$$\begin{aligned}
\sum_{t=0}^{T-1}[u(c_{t+1}) - u(\hat{c}_{t+1})] &\leq \sum_{t=0}^{T-1}\hat{p}_{t+1}(c_{t+1} - \hat{c}_{t+1}) \\
&= \sum_{t=0}^{T-1}\{[\hat{p}_{t+1}f(k_t) - \hat{p}_t k_t] - [\hat{p}_{t+1}f(\hat{k}_t) - \hat{p}_t \hat{k}_t]\} \\
&\quad + [\hat{p}_T \hat{k}_T - \hat{p}_T k_T] \\
&\leq 0
\end{aligned}$$

This means that $\langle \hat{k}_t \rangle$ is short-run optimal. The following corollary of Theorem 3 is then immediate.

Corollary 2. *Assume (U.1), (F.1) and (F.2). Let $\langle \hat{k}_t \rangle$ be a competitive program from $\mathbf{k} \in (0, \bar{k}]$, with associated prices $\langle \hat{p}_t \rangle$. Then $\langle \hat{k}_t \rangle$ is utilitarian maximal if and only if it is efficient.*

This characterization of utilitarian maximality is useful in constructing the example (in Section 5) which shows that the equivalence result (of Section 3) fails without the Phelps–Koopmans condition.

Neither Theorem 3 nor Corollary 2 depends on the restriction **(G)** or the Phelps–Koopmans condition, used in the analysis of Section 3.[11] However, if restriction **(G)** and the Phelps–Koopmans condition do hold (so that the equivalence result of Theorem 2 is valid), then Corollary 2 immediately provides the sufficiency part of Brock's (1971) characterization of weak maximality for this class of non-convex models: if a program is competitive and efficient, then it is weakly maximal. This result can be stated as follows.

Corollary 3. *Assume (U.1), (F.1), (F.2), (G) and the Phelps–Koopmans condition. Let $\langle \hat{k}_t \rangle$ be a competitive program from $\mathbf{k} \in (0, \bar{k}]$, with associated prices $\langle \hat{p}_t \rangle$. Then $\langle \hat{k}_t \rangle$ is weakly maximal if and only if it is efficient.*

5. On the role of the Phelps–Koopmans condition in the equivalence result

In Section 3, we showed that (under the restriction **(G)**), the Phelps–Koopmans condition is sufficient to ensure the equivalence of the set of weakly maximal programs and the set of utilitarian maximal programs. In this section, we show that if the Phelps–Koopmans condition does not hold, then the equivalence result fails; that is, we develop in detail an example in which a utilitarian maximal program exists, which is not weakly maximal.[12]

A key observation in constructing such an example is that if $\langle k_t \rangle$ is a weakly maximal program from $\mathbf{k} \in (0, \bar{k}]$, and k_t converges to k, then k must be the minimal golden-rule stock, k^*. In particular, in the class of aggregative models satisfying (F.1)-(F.2), (U.1) and (G), if $\langle k_t \rangle$ is a weakly maximal program from $\mathbf{k} \in (0, \bar{k}]$, then k_t converges to the minimal golden-rule stock, k^*, since convergence of k_t is assured by Proposition 2. This observation is of independent interest and is therefore stated and proved below.

Proposition 4. *Assume (U.1), (F.1) and (F.2). Let $\langle k_t \rangle$ be a weakly maximal program from $\mathbf{k} \in (0, \bar{k}]$.*

(i) *Suppose k_t converges to k as $t \to \infty$. Then $k = k^*$.*
(ii) *Suppose (G) holds. Then, k_t converges to k^* as $t \to \infty$.*

Proof. (i) Note first that since a good program exists from $\mathbf{k}$, a weakly maximal program $\langle k_t \rangle$ must be good. This implies that k must be a golden-rule stock. Thus, if $k \neq k^*$, we have $\alpha \equiv (k - k^*) > 0$. Denote $u[f(k^* + (\alpha/2)) - k^*] - u(c^*)$ by β; then $\beta > 0$. Since $\lim_{t\to\infty} k_t = k$, we can choose $N \in \mathbb{N}$, such that:

$$p^*|k_t - k| \leq \left(\frac{\beta}{3}\right) \quad \text{and} \quad |k_t - k| \leq \left(\frac{\alpha}{2}\right) \quad \text{for all} \quad t \geq N \tag{7}$$

[11] Indeed, the reader can check that the concavity of the utility function, u, also does not play any role in these two results.

[12] We would like to thank Debraj Ray for pointing us in the right direction in search of this example.

Then, for all $T > N+1$ we can write:

$$\begin{aligned}\sum_{t=N+1}^{T}[u(c_t)-u(c^*)] &\le \sum_{t=N+1}^{T} p^*(c_t-c^*)\\ &= \sum_{t=N+1}^{T} p^*\{[f(k_{t-1})-k_t]-[f(k)-k]\}\\ &= \sum_{t=N+1}^{T}\{[p^*f(k_{t-1})-p^*k_{t-1}]-[p^*f(k)-p^*k]\}\\ &\quad +[p^*k_N-p^*k]-[p^*k_T-p^*k]\\ &\le \left(\frac{2\beta}{3}\right)\end{aligned} \tag{8}$$

the first inequality in (8) following from (UP), and the last inequality in (8) following from (7) and (FP). Thus, denoting $\sum_{t=1}^{N}[u(c_t)-u(c^*)]$ by γ, we have for all $T > N+1$,

$$\sum_{t=1}^{T}[u(c_t)-u(c^*)] \le \gamma + \left(\frac{2\beta}{3}\right) \tag{9}$$

Let $\langle k'_t\rangle$ be a sequence defined by: $k'_t = k_t$ for $t = 0, \ldots, N$, and $k'_t = k^*$ for $t \ge N+1$. Note that $k'_N = k_N = (k_N - k) + (k - k^*) + k^* \ge -(\alpha/2) + \alpha + k^* = k^* + (\alpha/2)$. Thus, $\langle k'_t\rangle$ is a program from $\mathbf{k}$ and for all $T > N+1$,

$$\begin{aligned}\sum_{t=N+1}^{T}[u(c'_t)-u(c^*)] &= u(c'_{N+1})-u(c^*)\\ &= u(f(k'_N)-k^*)-u(c^*)\\ &\ge u(f(k^*+(\alpha/2))-k^*)-u(c^*)=\beta\end{aligned}$$

so that:

$$\begin{aligned}\sum_{t=1}^{T}[u(c'_t)-u(c^*)] &= \sum_{t=1}^{N}[u(c'_t)-u(c^*)]+\sum_{t=N+1}^{T}[u(c'_t)-u(c^*)]\\ &\ge \sum_{t=1}^{N}[u(c_t)-u(c^*)]+\beta=\gamma+\beta\end{aligned} \tag{10}$$

Thus, for all $T > N+1$, using (9) and (10),

$$\sum_{t=1}^{T}[u(c'_t)-u(c_t)] \ge \left(\frac{\beta}{3}\right) \tag{11}$$

contradicting the weak maximality of $\langle k_t\rangle$.

(ii) Since $\langle k_t\rangle$ is good, and (G) holds, k_t must converge to some golden-rule stock by Proposition 2, and so k_t converges to k^* by (i). □

The result of Proposition 4 entails that all we need to provide in the rest of this section is an example in which a utilitarian maximal program exists and converges to a golden-rule stock higher than the minimal golden-rule stock. Proposition 4 then implies that the equivalence between weak maximality and utilitarian maximality fails. And, Proposition 3 implies that the Phelps–Koopmans condition fails.

The production function in our example has two golden-rule stocks. We construct a competitive program for which the sequence of stocks converges to the higher golden-rule stock (from above), but is nevertheless efficient. By Corollary 2 in the previous section, it is utilitarian maximal.

We observe that this construction is harder than the construction of an efficient program for which the sequence of stocks converges to a golden-rule stock higher than the minimum golden-rule stock (a violation of the Phelps–Koopmans condition), the additional difficulty arising from the fact that we have to ensure that the program also satisfies the Ramsey–Euler equation at each date.

The construction of the example involves “reverse engineering”. We first choose a sequence of stocks that will be suitable to work with. We then specify the production function (with two golden-rule stocks) such that this sequence of stocks is a program, which converges to the higher golden-rule stock. Finally, we specify the utility function which (together with the specification of the production function) makes the chosen program a competitive program. The steps of the formal analysis are somewhat involved and have been divided into five steps for clarity.

Step 1. [*Construct a monotone decreasing sequence from* $\mathbf{k}=4$]: let $m \equiv \sqrt{2}$. Define a sequence $\langle k_t \rangle$ by

$$k_{t+1} = mk_t^{1/2} \quad \text{for} \quad t \geq 0; \quad k_0 = \mathbf{k} = 4 \tag{12}$$

The sequence is well-defined by (12). It has the following properties:

$$(i)\ k_t > 2 \quad \text{for} \quad t \geq 0 \qquad (ii)\ k_{t+1} < k_t \quad \text{for} \quad t \geq 0 \tag{13}$$

Clearly, we have $k_t > 0$for $t \geq 0$. To check (i), note that $k_0 > 2$, and if $k_t > 2$, then $k_{t+1} = mk_t^{1/2} > m\sqrt{2} = 2$, so that the property follows by induction. For (ii), note that $(k_{t+1}/k_t) = m/k_t^{1/2} < m/\sqrt{2} = 1$, the inequality following from property (i).

Thus, $\langle k_t \rangle$ is a decreasing sequence, bounded below by 2, so it converges to some k, and using (12), it is easy to check that $k = 2$.

Step 2. [*Define a production function suitably so that* $\langle k_t \rangle$*, defined in Step 1, is a program from* $\mathbf{k}=4$]: to this end, let $a = 2\sqrt{2}$, and define $f : [0, 8] \to [0, 8]$, by

$$f(x) = \begin{cases} 3x & \text{for } 0 \leq x \leq 1 \\ 3 + (x-1)^2 & \text{for } 1 < x \leq 2 \\ ax^{1/2} & \text{for } 2 < x \leq 8 \end{cases} \tag{14}$$

One can then satisfy (F.1) and (F.2) by defining $f(x) = 8 + (1/2)(x - 8)$ for all $x > 8$. Note that $\mathcal{G} = \{1, 2\}$, hence $k^* = 1$, $c^* = 2$ and $\bar{k} = 8$. We will focus our attention on stocks in [0, 8].

Define $s = 1/2$. For $t \geq 0$, we have $f(k_t) - k_{t+1} = ak_t^{1/2} - k_{t+1} = 2(sa)k_t^{1/2} - k_{t+1} = 2mk_t^{1/2} - k_{t+1} = k_{t+1}$, by (12). Thus, $\langle k_t \rangle$ is a program from $\mathbf{k} = 4$, and $c_{t+1} = k_{t+1}$ for all $t \geq 0$.

Step 3. [*Define the utility function suitably so that* $\langle k_t \rangle$ *satisfies the Ramsey-Euler equations*]: let $u : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$u(c) = \begin{cases} 2c^{1/2} - 2 & \text{for } 0 \leq c \leq 1 \\ \ln c & \text{for } c > 1 \end{cases}$$

Clearly, u satisfies (U.1).

Since $c_t = k_t > 2$ for all $t \geq 1$, we have $u'(c_t) = (1/c_t) = (1/k_t)$ for all $t \geq 1$. And, since $k_t > 2$ for all $t \geq 1$, we have $f'(k_t) = (1/2)a/k_t^{1/2} = m/k_t^{1/2}$ for all $t \geq 1$. Thus, for all $t \geq 1$, we have, by using (12):

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{m}{k_t^{1/2}} = f'(k_t) \tag{RE}$$

so that the Ramsey–Euler equations are satisfied.

Step 4. [*Define a sequence* $\langle p_t \rangle$, such that $\langle k_t \rangle$ *is a competitive program from* $\mathbf{k}$*, with associated prices*$\langle p_t \rangle$]: let us define:

$$p_t = u'(c_t) = \left(\frac{1}{c_t}\right) \quad \text{for } t \geq 1; p_0 = p_1 f'(\mathbf{k}) \tag{P}$$

Since u is concave, and u is differentiable at each c_t, we have, for each $t \geq 1$,

$$u(c) - u(c_t) \leq u'(c_t)(c - c_t) = p_t(c - c_t) \quad \text{for all} \quad c \geq 0$$

verifying (CE)(a).

It remains to verify (CE)(b). To this end, define $g : [0, 8] \to [0, 8]$ by

$$g(x) = \begin{cases} 3x & \text{for } 0 \leq x \leq 1 \\ 3 + (x-1) & \text{for } 1 < x \leq 2 \\ ax^{1/2} & \text{for } 2 < x \leq 8 \end{cases}$$

Note that $g(x) = f(x)$ for $x \in [0, 1]$ and $x \in [2, 8]$, and $g(x) > f(x)$ for $x \in (1, 2)$. Also, g is a *concave* function on [0, 8], since the right-hand derivative of g is non-increasing on [0, 8), and g is continuous on [0, 8].

For each $t \geq 1$, we have $k_t \in (2, 4]$, and so g is differentiable at each k_t. This yields:

$$f(x) - f(k_t) \leq g(x) - g(k_t) \leq g'(k_t)(x - k_t) = f'(k_t)(x - k_t) \quad \text{for all } x \geq 0$$

so that for each $t \geq 1$,

$$p_{t+1}[f(x) - f(k_t)] \leq p_{t+1} f'(k_t)(x - k_t) = p_t(x - k_t) \quad \text{for all } x \geq 0 \tag{15}$$

by using (RE) and (P). Also, since $\mathbf{k} = 4$, we have:

$$f(x) - f(\mathbf{k}) \leq g(x) - g(\mathbf{k}) \leq g'(\mathbf{k})(x - \mathbf{k}) = f'(\mathbf{k})(x - \mathbf{k}) \quad \text{for all } x \geq 0$$

This yields:

$$p_1[f(x) - f(\mathbf{k})] \le p_1 f'(\mathbf{k})(x - \mathbf{k}) = p_0(x - \mathbf{k}) \quad \text{for all } x \ge 0 \tag{16}$$

by using (P). Clearly, (15) and (16) verify (CE)(b).

Step 5. [*Show that* $\langle k_t \rangle$ *is efficient*]: suppose, on the contrary, there is a feasible path $\{k'_t\}$ from $\mathbf{k} = 4$, such that:

$$c'_{t+1} \ge c_{t+1} \quad \text{for all } t \ge 0 \tag{17}$$

with strict inequality in (17) for some $t = \tau \ge 0$. Denoting the difference between the two sides of (17) for $t = \tau$ by α, we have $p_{\tau+1}\alpha \le \sum_{t=0}^{T} p_{t+1}(c'_{t+1} - c_{t+1}) = \sum_{t=0}^{T}(p_{t+1}f(k'_t) - p_t k'_t) - (p_{t+1}f(k_t) - p_t k_t) + p_{T+1}(k_{T+1} - k'_{T+1})$ for all $T \ge \tau$. Thus, by (15), $p_{T+1}(k_{T+1} - k'_{T+1}) \ge p_{\tau+1}\alpha > 0$ for all $T \ge \tau$. So, we have $(k_{T+1} - k'_{T+1}) > 0$ for all $T \ge \tau$, and since $p_t = u'(c_t) = 1/c_t = 1/k_t < 1/2$ for all $t \ge 1$, we obtain:

$$(k_{T+1} - k'_{T+1}) \ge 2p_{\tau+1}\alpha \equiv \beta \quad \text{for all } T \ge \tau \tag{18}$$

Since $k_t \to 2$ as $t \to \infty$, (18) implies that there is $N > \tau$, such that $k'_t < 2$ for all $t \ge N$.

We focus now on $t \ge N$. For such t, we have:

$$k'_{t+1} = f(k'_t) - c'_{t+1} \le 2 + k'_t - c_{t+1} = 2 + k'_t - k_{t+1} < k'_t \tag{19}$$

Thus, k'_t is decreasing over time for $t \ge N$, and (since it is bounded below) must converge to some $k' \in [0, 2)$. In this case, c'_{t+1} must converge to $f(k') - k'$. But, by (17), we must then have $f(k') - k' \ge 2$. There is only one value of $x \in [0, 2)$ for which this is true, namely $k^* \equiv 1$. Thus, k'_t is decreasing over time for $t \ge N$ and converging to $k^* \equiv 1$.

For $t \ge N$, we denote $[k'_t - 1]$ by ε_t. Then, we have for $t \ge N$, using (17) and (14), $k'_{t+1} = f(k'_t) - c'_{t+1} \le 3 + \varepsilon_t^2 - 2 - \gamma_{t+1}$, where $\gamma_{t+1} \equiv c_{t+1} - 2 = k_{t+1} - 2 > 0$ for $t \ge 0$. Thus, we must have:

$$\varepsilon_{t+1} \le \varepsilon_t^2 - \gamma_{t+1} \quad \text{for all } t \ge N' \tag{20}$$

We now focus on the sequence $\{\gamma_{t+1}\}$. Clearly, by concavity of f on $[2, 8]$, we have:

$$(k_{t+1} - k) = mk_t^{1/2} - mk^{1/2} \ge \left[\frac{m}{2k_t^{1/2}}\right](k_t - k) > (\frac{1}{4})(k_t - k) \quad \text{for all } t \ge N$$

So, for all $t \ge N$,

$$\frac{\gamma_{t+1}}{\gamma_t} = \frac{c_{t+1} - 2}{c_t - 2} = \frac{k_{t+1} - 2}{k_t - 2} = \frac{k_{t+1} - k}{k_t - k} > \left(\frac{1}{4}\right) \tag{21}$$

Since $\varepsilon_t \to 0$ as $t \to \infty$, we can find $N' > N$ such that $\varepsilon_t < (1/8)$ for all $t \ge N'$. Then, for $t \ge N'$, using (20) and (21), we obtain:

$$\begin{aligned}\frac{\varepsilon_{t+1}}{\gamma_{t+1}} &\le \frac{\varepsilon_t^2}{\gamma_{t+1}} - 1 \le \frac{4\varepsilon_t^2}{\gamma_t} - 1 = (4\varepsilon_t)\left[\frac{\varepsilon_t}{\gamma_t}\right] - 1 \\ &\le \left(\frac{1}{2}\right)\left[\frac{\varepsilon_t}{\gamma_t}\right] - 1 \\ &\le \left(\frac{1}{2}\right)\left[\frac{\varepsilon_t}{\gamma_t}\right]\end{aligned} \tag{22}$$

Then, by the last inequality of (22), we obtain $(\varepsilon_t/\gamma_t) \to 0$ as $t \to \infty$. And, using this information in the last but one inequality of (22), we must have $(\varepsilon_{t+1}/\gamma_{t+1}) < 0$ for all large t, a contradiction. This establishes our claim that $\langle k_t \rangle$ is an efficient path from $\mathbf{k}$.

6. Proofs

Proof of Proposition 1. Denote $u'_+(c^*)$ by p^*; p^* is well-defined since $c^* > 0$. Since u is strictly increasing and concave, we must have $p^* > 0$. By concavity of u we have $u(c) - u(c^*) \le u'_+(c^*)(c - c^*) = p^*(c - c^*)$ for all $c \ge 0$. By transposing terms (UP) can be easily verified.

For any $k \in \mathcal{G}$, we have $f(k) - k \ge f(x) - x$ for all $x \ge 0$. Multiplying the inequality throughout by $p^* > 0$ yields (FP). □

Remark 1. The inequality is strict in (UP) when $c \ne c^*$. This follows from the strict concavity of u. Also (FP) holds with strict inequality whenever $x \notin \mathcal{G}$.

Pick $k'' \in (k^*, \bar{k})$, with k'' sufficiently close to $\bar{k}$ so that $f(\bar{k}) - k'' \le c^*$.

Lemma 1. *If $\langle k_t \rangle$ is an efficient program from $\mathbf{k} \in (0, \bar{k}]$, then there is a subsequence $(t_1, t_2, \ldots\ldots)$ such that $k_{t_s} \leq k''$ for all $s \in \mathbb{N}$.*

Proof. If the Lemma is not true then there is some $N \in \mathbb{N}$, such that $k_t > k''$ for all $t \geq N$. In this case, $c_t = f(k_{t-1}) - k_t < f(\bar{k}) - k'' \leq c^*$ for all $t > N$. Defining $k'_t = k_t$ for $t = 0, 1, \ldots, N$ and $k'_t = k^*$ for $t > N$, we have $c'_t = c_t$ for $t = 1, \ldots, N$ and $c'_t \geq c^* > c_t$ for $t > N$. This contradicts the efficiency of $\langle k_t \rangle$. □

For $x \in (0, \bar{k}]$, define $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$ for all $n \in \mathbb{N}$. Then, $\langle f^n(x) \rangle_{n=0}^{\infty}$ is a non-decreasing sequence, which converges to $\bar{k}$. Thus, for every $\tilde{k} \in (0, \bar{k})$,

$$i(x, \tilde{k}) = \min\{i \in \mathbb{N} : f^i(x) \geq \tilde{k}\}$$

is well defined.

Proof of Theorem 1. Given $\mathbf{k} \in (0, \bar{k}]$, denote $i(\mathbf{k}, k^*)$ by M and $i(k^*, k'')$ by N. Since $\langle k_t \rangle$ is utilitarian maximal, it is efficient and so by Lemma 1, there is a subsequence $(t_1, t_2, \ldots)$ such that $k_{t_s} \leq k''$ for all $s \in \mathbb{N}$. Pick any $s \in \mathbb{N}$ such that $t_s > M + N + 1$. Define a sequence $\langle k'_t \rangle$ as follows:

$$\left.\begin{array}{l} (i)\ (k'_0, \ldots, k'_{M-1}) = (f^0(\mathbf{k}), \ldots, f^{M-1}(\mathbf{k})) \\ (ii)\ (k'_M, \ldots, k'_{t_s-N-1}) = (k^*, \ldots, k^*) \\ (iii)\ (k'_{t_s-N}, \ldots, k'_{t_s}) = (f^0(k^*), \ldots, f^{N-1}(k^*), k_{t_s}) \\ (iv)\ k'_t = k_t \quad \text{for all } t > t_s \end{array}\right\}$$

It is straightforward to check that $\langle k'_t \rangle$ is a program from $\mathbf{k}$, with $c'_t \geq 0$ for $t = 1, \ldots, M$; $c'_t = c^*$ for $t = M+1, \ldots, t_s - N$; $c'_t \geq 0$ for $t = t_s - N + 1, \ldots, t_s$ and $c'_t = c_t$ for $t > t_s$. Clearly,

$$\sum_{t=1}^{t_s} (u(c'_t) - u(c^*)) \geq -(M+N)[u(c^*) - u(0)]$$

Since $\langle k_t \rangle$ is utilitarian maximal, and $c'_t = c_t$ for $t > t_s$, we must have:

$$\sum_{t=1}^{t_s} (u(c_t) - u(c^*)) \geq -(M+N)[u(c^*) - u(0)]$$

Since this inequality must hold for each $s \in \mathbb{N}$ satisfying $t_s > M + N + 1$, $\langle k_t \rangle$ cannot be bad. By Corollary 1, it must be good. □

Let us define

$$\alpha(c) = [u(c^*) - p^* c^*] - [u(c) - p^* c] \quad \text{for all } c \geq 0$$

and

$$\beta(x) = p^*[f(k^*) - k^*] - p^*[f(x) - x] \quad \text{for all } x \geq 0$$

By (UP), $\alpha(c) \geq 0$ for all $c \geq 0$ and by (FP), $\beta(x) \geq 0$ for all $x \geq 0$. For any feasible program $\langle k_t \rangle$ from $\mathbf{k} \in [0, \bar{k}]$ the following identity can be easily verified:

$$\sum_{t=1}^{T} [u(c^*) - u(c_t)] = p^*[f(k_T) - f(\mathbf{k})] + \sum_{t=1}^{T} \alpha_t + \sum_{t=1}^{T} \beta_t \tag{IG}$$

where, $\alpha_t = \alpha(c_t)$ and $\beta_t = \beta(k_t)$ and $\langle c_t \rangle$ is the consumption sequence associated with $\langle k_t \rangle$.

Lemma 2.

(i) *If $\langle k_t \rangle$ is a good program, then the sequence $(f(k_t) - k_t)$ must converge to c^* as $t \to \infty$.*
(ii) *If $\langle k_t \rangle$ is a good program and $\langle c_t \rangle$ is the the consumption sequence associated with $\langle k_t \rangle$, then c_t must converge to c^* as $t \to \infty$.*

Proof. *Since $\langle k_t \rangle$ is a good program, there is some $G \in \mathbb{R}$ such that:*

$$G \geq \sum_{t=1}^{T} [u(c^*) - u(c_t)] \geq -p^* f(\bar{k}) + \sum_{t=1}^{T} \alpha_t + \sum_{t=1}^{T} \beta_t \quad \text{for all } T \geq 1 \tag{23}$$

using (IG). The identity (23) implies that the partial sums $\sum_{t=1}^{T}\alpha_t$ and $\sum_{t=1}^{T}\beta_t$ are bounded above. Since $\alpha_t \geq 0$ and $\beta_t \geq 0$ for all t, the partial sums $\sum_{t=1}^{T}\alpha_t$, $\sum_{t=1}^{T}\beta_t$ are non-decreasing. Hence, $\sum_{t=1}^{\infty}\alpha_t$ and $\sum_{t=1}^{\infty}\beta_t$ are convergent series, and $\alpha_t \to 0$ and $\beta_t \to 0$ as $t \to \infty$. Since $\beta_t \to 0$ as $t \to \infty$, (i) is established.

Since $\alpha_t \to 0$ as $t \to \infty$, we have $[u(c_t) - p^*c_t] \to [u(c^*) - p^*c^*]$. We now claim that $c_t \to c^*$ as $t \to \infty$. For if this is not the case, then since $c_t \in [0, \bar{k}]$ for all $t \geq 1$, there is a convergent subsequence $\langle c_{t_s}\rangle$ of $\langle c_t\rangle$ which converges to $c \neq c^*$. Since $[u(c_t) - p^*c_t] \to [u(c^*) - p^*c^*]$ as $t \to \infty$, $[u(c_{t_s}) - p^*c_{t_s}]$ must also converge to $[u(c^*) - p^*c^*]$. However, by continuity of u, $[u(c_{t_s}) - p^*c_{t_s}]$ converges to $[u(c) - p^*c]$ and $[u(c) - p^*c] < [u(c^*) - p^*c^*]$ since $c \neq c^*$ (by strict concavity of u). This contradiction proves (ii). □

Proof of Proposition 2. Observe using Lemma 4, since $\langle k_t \rangle$ is a good program,

$$(f(k_t) - k_t) \to c^* \quad \text{as } t \to \infty \tag{24}$$

and

$$(f(k_t) - k_{t+1}) \to c^* \quad \text{as } t \to \infty \tag{25}$$

We would like to show, using (24) and (25), that $k_t \to k'$ where k' is *some* golden-rule stock. Let us write $\mathcal{G} = \{k^1, \ldots, k^n\}$, with

$$0 < k^1 < \cdots < k^n < \bar{k}$$

Define:

$$\theta = \min\{k^1, k^2 - k^1, k^3 - k^2, \ldots, k^n - k^{n-1}, \bar{k} - k^n\}$$

Since $(f(k^i) - k^i) = c^*$ for $i \in \{1, \ldots, n\}$, and f is continuous, we know that, for each $k^i \in \mathcal{G}$,

$$c^* - (f(x) - x) \to 0 \quad \text{as } x \to k^i$$

Thus, we can find $\alpha \in (0, \theta/4)$ with α sufficiently close to zero so that for each $k^i \in \mathcal{G}$,

$$c^* - (f(x) - x) < \frac{\theta}{4} \quad \text{for all } x \in [k^i - \alpha, k^i + \alpha] \tag{26}$$

Note that the finiteness of the set $\mathcal{G}$ is used to obtain (26). Define for each $i \in \{1, \ldots, n\}$,

$$A^i = (k^i - \alpha, k^i + \alpha), \bar{A}^i = [k^i - \alpha, k^i + \alpha]$$

and

$$A = \cup_{i=1}^{n} A^i; \ \bar{A} = \cup_{i=1}^{n} \bar{A}^i; \ B = [0, \bar{k}] \sim A$$

Clearly, A is open and B is a non-empty, closed and bounded set. Define:

$$\lambda = \min_{x \in B}\{c^* - (f(x) - x)\} \tag{27}$$

and

$$\eta = \max_{x \in \bar{A}}\{c^* - (f(x) - x)\} \tag{28}$$

Note that $\lambda > 0$ since B contains no golden-rule stock; also, $\eta > 0$ since $\bar{A}$ contains points other than golden-rule stocks. Further, $\eta < (\theta/4)$ by (26). Denote $\min\{\lambda, \eta\}$ by μ. Using (24) and (25), choose $N \in \mathbb{N}$ such that for all $t \geq N$,

$$\{c^* - (f(k_t) - k_t)\} < \left(\frac{\mu}{2}\right) \tag{29}$$

and

$$|c^* - (f(k_t) - k_{t+1})| < \left(\frac{\mu}{2}\right) \tag{30}$$

Since $\mu/2 < \lambda$, (27) and (29) imply that, for each $t \geq N$, we have $k_t \notin B$. That is,

$$k_t \in A \quad \text{for each } t \geq N \tag{31}$$

Clearly, (31) implies that there is some $r \in \{1, \ldots, n\}$, such that $k_N \in A^r$. We now claim that:

$$k_t \in A^r \quad \text{for all } t \geq N \tag{32}$$

If claim (32) were false, let $T > N$ be the first period where it fails to hold. Then, $k_{T-1} \in A^r$, but $k_T \notin A^r$. By (31), we can find $s \in \{1, \ldots, n\}$, such that $k_T \in A^s$; clearly $s \neq r$. Since $k_{T-1} \in A^r$, we have $k_{T-1} \in \bar{A}$, and by (28), $c^* - [f(k_{T-1}) - k_{T-1}] \leq \eta$, so that:

$$[f(k_{T-1}) - k_{T-1}] \geq c^* - \eta \tag{33}$$

Since $k_{T-1} \in A^r$, but $k_T \in A^s$, $s \neq r$, we have the following two possibilities: Case (i) $[k_{T-1} - k_T] > (\theta/2)$; Case (ii) $[k_{T-1} - k_T] < -(\theta/2)$. In case (i), we get, using (33),

$$\begin{aligned} f(k_{T-1}) - k_T &= [f(k_{T-1}) - k_{T-1}] + [k_{T-1} - k_T] \\ &> [f(k_{T-1}) - k_{T-1}] + \left(\frac{\theta}{2}\right) \\ &\geq c^* - \eta + \left(\frac{\theta}{2}\right) > c^* + \left(\frac{\theta}{4}\right) > c^* + \mu \end{aligned} \tag{34}$$

the last line of (34) using the fact that $\mu \leq \eta < (\theta/4)$. But, (33) clearly contradicts (30). In case (ii), we get:

$$\begin{aligned} f(k_{T-1}) - k_T &= [f(k_{T-1}) - k_{T-1}] + [k_{T-1} - k_T] \\ &< [f(k_{T-1}) - k_{T-1}] - \left(\frac{\theta}{2}\right) \\ &\leq c^* - \left(\frac{\theta}{2}\right) < c^* - \mu \end{aligned} \tag{35}$$

the last line of (35) using the fact that $\mu \leq \eta < (\theta/4) < (\theta/2)$. But, (35) clearly contradicts (30). This establishes the claim (32). Since k^r is the only value in A^r at which $f(x) - x = c^*$, (24) and (32) imply that $k_t \to k^r$ as $t \to \infty$. □

Proof of Theorem 2. A weakly maximal program is clearly utilitarian maximal. It remains to establish the converse. Let $\langle k_t \rangle$ be a utilitarian maximal program from $\mathbf{k} \in (0, \bar{k}]$. Suppose $\langle k_t \rangle$ is not weakly maximal from $\mathbf{k}$. Then there is a feasible program $\langle k'_t \rangle$ from $\mathbf{k}$, $N' \in \mathbb{N}$ and $\alpha > 0$ such that for all $T > N'$,

$$\sum_{t=1}^{T} (u(c'_t) - u(c_t)) \geq \alpha \tag{36}$$

Since $\langle k_t \rangle$ is a good program by Theorem 1, there exists some $G \in \mathbb{R}$ and some $\bar{T} \in \mathbb{N}$ such that for all $T \geq \bar{T}$,

$$\sum_{t=1}^{T} (u(c_t) - u(c^*)) \geq G \tag{37}$$

Then for any $T > \max\{\bar{T}, N'\}$, using (36) and (37),

$$\begin{aligned} \sum_{t=1}^{T} (u(c'_t) - u(c^*)) &= \sum_{t=1}^{T} (u(c'_t) - u(c_t)) + \sum_{t=1}^{T} (u(c_t) - u(c^*)) \\ &\geq \alpha + G \end{aligned} \tag{38}$$

This shows that the program $\langle k'_t \rangle$ is a good program.

From Proposition 2, we know that there is some $k' \in \mathcal{G}$ such that $k'_t \to k'$ as $t \to \infty$. By Proposition 3, $k_t \to k^* \leq k'$ as $t \to \infty$. Since u and f are continuous, there exists $M \in \mathbb{N}$ with $M > N'$ such that $f(k'_M) - k_{M+1} \geq 0$, and

$$u(f(k'_M) - k_{M+1}) - u(f(k_M) - k_{M+1}) \geq -\left(\frac{\alpha}{2}\right) \tag{39}$$

Let us now define a sequence $\langle k''_t \rangle$ as follows: $k''_t = k'_t$ for all $t = 1,...,M$ and $k''_t = k_t$ for all $t > M$. Clearly $\langle k''_t \rangle$ is a program from $\mathbf{k}$ and $u(c''_t) = u(c_t)$ for all $t \geq M + 2$. Also,

$$\begin{aligned} \sum_{t=1}^{M+1} (u(c''_t) - u(c_t)) &= \sum_{t=1}^{M} (u(c''_t) - u(c_t)) + (u(c''_{M+1}) - u(c_{M+1})) \\ &= \sum_{t=1}^{M} (u(c'_t) - u(c_t)) + (u(c''_{M+1}) - u(c_{M+1})) \\ &\geq \alpha - \left(\frac{\alpha}{2}\right) = \left(\frac{\alpha}{2}\right) > 0 \end{aligned} \tag{40}$$

The second line in (40) follows from noting that $k''_t = k'_t$ for all $t = 1, \ldots, M$. The first term in the inequality in the last line of (40) follows from (36) and the fact that $M > N'$. The second term in the inequality in the last line of (40) follows from (39). This shows that $\langle k''_t \rangle$ utilitarian dominates $\langle k_t \rangle$, a contradiction. □

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References

Asheim, G.B., Tungodden, B., 2004. Resolving distributional conflicts between generations. Economic Theory 24, 221–230.
Atsumi, H., 1965. Neoclassical growth and the efficient program of capital accumulation. Review of Economic Studies 32, 127–136.
Basu, K., Mitra, T., 2007. Utilitarianism for infinite utility streams: a new welfare criterion and its axiomatic characterization. Journal of Economic Theory 133, 350–373.
Brock, W.A., 1970a. On existence of weakly maximal programmes in a multi-sector economy. Review of Economic Studies 37, 275–280.
Brock, W.A., 1970b. An axiomatic basis for the Ramsey–Weizsäcker overtaking criterion. Econometrica 38, 927–929.
Brock, W.A., 1971. Sensitivity of optimal growth paths with respect to a change in target stocks. In: Bruckmann, G., Weber, W. (Eds.), Contributions to the von Neumann Growth Model. Springer Verlag, pp. 73–89.
Clark, C.W., 1971. Economically optimal policies for the utilization of biologically renewable resources. Mathametical Biosciences 12, 245–260.
Gale, D., 1967. On optimal development in a multi-sector economy. Review Economic Studies 34, 1–18.
Majumdar, M., Mitra, T., 1982. Intertemporal allocation with a non-convex technology: the aggregative framework. Jouranl of Economic Theory 27, 101–136.
Majumdar, M., Mitra, T., 1983. Dynamic optimization with a non-convex technology: the case of a linear objective function. Review Economic Studies 50, 143–151.
McKenzie, L.W., 1968. Accumulation programs of maximum utility and the von Neumann facet. In: Wolfe, J.N. (Ed.), Value, Capital and Growth. Edinburgh University Press, pp. 353–383.
Mitra, T., 1992. On the existence of a stationary stock for a multi-sector economy with non-convex technology. In: Majumdar, M. (Ed.), Equilibrium and Dynamics: Essays in Honor of David Gale. Macmillan, London, pp. 214–240.
Mitra, T., Ray, D., 1984. Dynamic optimization on a nonconvex feasible set: some general results for non-smooth technologies. Journal of Economics 44, 151–175.
Mitra, T., Ray, D., 2009. On the Phelps–Koopmans theorem. Journal of Economic Theory, doi:10.1016/j.jet.2009.08.004.
Phelps, E.S., 1962. The end of the golden age in Solovia: comment. American Economic Review 52, 1097–1099.
Phelps, E., 1965. Second essay on the golden rule of accumulation. American Economic Review 55, 793–814.
Radner, R., 1961. Paths of economic growth that are optimal with regard only to final states: a turnpike theorem. Review of Economic Studies 28, 98–104.
Ramsey, F.P., 1928. A mathematical theory of savings. Economic Journal 38, 543–559.
von Weizšacker, C.C., 1965. Existence of optimal programs of accumulation for an infinite time horizon. Review of Economic Studies 32, 85–104.